COMPLEX ANALYTIC PROJECTIVE CONNECTIONS

ALEXANDRU MIHAI

Let M be a complex analytic manifold of dimension $n \geqslant 2$. *A complex analytic projective connection* on M is defined with respect to a coordinate covering (U, z^i) of M by its components (U, Γ^i_{jk}), which are complex analytic functions satisfying the compatibility relations

$$\frac{\partial z^i}{\partial z'^\alpha} \cdot \frac{\partial^2 z'^\alpha}{\partial z^j \partial z^k} - \frac{1}{n+1} \delta^i_j \frac{\partial \log \Delta}{\partial z^k} - \frac{1}{n+1} \delta^i_k \frac{\partial \log \Delta}{\partial z^j}$$

$$= \Gamma^i_{jk} - \Gamma'^\alpha_{\rho\gamma} \frac{\partial z^i}{\partial z'^\alpha} \cdot \frac{\partial z'^\beta}{\partial z^j} \cdot \frac{\partial z'^\gamma}{\partial z^k},$$

whenever $U \cap U' \neq \varnothing$, where $\Delta = \det(\partial z'^i / \partial z^j)$, [2, p. 99].

The left side determines a class $h(M) \in H^1(M, \mathbf{T^*} \otimes \mathbf{T} \otimes \mathbf{T^*})$, where $\mathbf{T}$ and $\mathbf{T^*}$ denote the sheaves of germs of cross sections of the tangent bundle T and the cotangent bundle T^* of M this is the obstruction to the existence of complex analytic projective connections on M.

One has

$$h(M) = a(T) - \frac{1}{n+1} I \cup a(\Lambda^n T) - \frac{1}{n+1} a(\Lambda^n T) \cup I^*,$$

where $a(T) \in H^1(M, \mathbf{T^*} \otimes \mathbf{T} \otimes \mathbf{T^*})$, $a(\Lambda^n T) \in H^1(M, \mathbf{T^*})$ are the Atiyah classes [1, p. 188], and $I \in H^0(M, \mathbf{T^*} \otimes \mathbf{T})$, $I^* \in H^0(M, \mathbf{T} \otimes \mathbf{T^*})$ are the identity endomorphisms, and "$\cup$" denotes the cup product.

The corresponding class in the differential case is always zero. The same is true if M is a Stein manifold.

If M is a compact Kähler manifold, $a(T)$ generates under the operation of the invariant polynomials of $GL_n(\mathbf{C})$, the characteristic cohomology ring of M (with complex coefficients) [1, Theorem 3]. Similarly, $h(M)$ will generate a ring which we will call *the projective characteristic ring of M*.

Theorem. *Let M be a compact Kähler manifold of dimension $n \geqslant 2$. Then the projective characteristic ring of M is generated by the following classes:*

$$h_0(M) = 1,$$

(∗)
$$h_j(M) = \sum_{k=0}^{j} \frac{(-1)^k}{k! \, (n+1)^k} ch_{j-k}(M) ch_1^k(M)$$

$$+ \frac{(-1)^j}{j! \, (n+1)^j} ch_1^j(M), \quad 2 \leqslant j \leqslant n,$$

where $ch_j(M)$ are the components of the Chern character of M.

Proof. Let Γ_{jk}^i be the components of the canonical linear connection of the tangent bundle T associated to the hermitian structure of M. The forms $R^i_{jkl} \, dz^k \wedge d\bar{z}^l$, $R_{kl} \, dz^k \wedge d\bar{z}^l$, where $R^i_{jkl} = \partial \Gamma_{jk}^i / \partial z^l$, and $R_{kl} = R^i_{ikl}$, represent the Atiyah classes $a(T)$ and $a(\Lambda^n T)$ by the Serre-Dolbeault isomorphism. Also

$$R^i_{jkl} \, dz^k \wedge d\bar{z}^l - \frac{1}{n+1} \delta_j^i R_{kl} \, dz^k \wedge d\bar{z}^l - \frac{1}{n+1} \delta_k^i R_{jl} \, dz^k \wedge d\bar{z}^l$$

represents the class $h(M)$.

It is well-known that the ring of invariant polynomials of $GL_n(\mathbb{C})$ is generated by $(1/j!) \, \mathrm{tr}(A^j)$. Consequently, the projective characteristic ring of M is generated by the classes $h_j(M)$ represented by the forms

$$\frac{1}{j!} \left(R^l_{l_2 k_1 k_2} \, dz^{k_1} \wedge d\bar{z}^{k_2} - \frac{1}{n+1} \delta_{l_2}^{l_1} R_{k_1 k_2} \, dz^{k_1} \wedge d\bar{z}^{k_2} \right.$$

$$- \frac{1}{n+1} \delta_{k_1}^{l_1} R_{l_2 k_2} \, dz^{k_1} \vee d\bar{z}^{k_2} \right) \wedge \cdots \wedge \left(R^{l_j}_{l_1 k_{2j-1} k_{2j}} \, dz^{k_{2j-1}} \wedge d\bar{z}^{k_{2j}} \right.$$

$$\left. - \frac{1}{n+1} \delta_{l_1}^{l_j} R_{k_{2j-1} k_{2j}} \, dz^{k_{2j-1}} \wedge d\bar{z}^{k_{2j}} - \frac{1}{n+1} \delta_{k_{2j-1}}^{l_j} R_{l_1 k_{2j}} \, dz^{k_{2j-1}} \wedge d\bar{z}^{k_{2j}} \right).$$

We recall that in the Kählerian case $R^i_{jkl} = R^i_{kjl}$. Then the formulas (∗) follows by standard calculations, since the form

$$\frac{(\sqrt{-1})^j}{j! \, (2\pi)^j} R^{l_1}_{l_2 k_1 k_2} \cdots R^{l_j}_{l_1 k_{2j-1} k_{2j}} \, dz^{k_1} \wedge d\bar{z}^{k_2} \wedge \cdots \wedge dz^{k_{2j-1}} \wedge d\bar{z}^{k_{2j}}$$

represents the j-the component of the Chern character of M.

Corollary. *If M admits a complex analytic projective connection, then $h_j(M) = 0, 2 \leqslant j \leqslant n$.*

For example, there is not such a connection on the product $\mathbf{P}^1\mathbf{C}$ $\times \cdots \times \mathbf{P}^1\mathbf{C}$ of $n \geqslant 2$ projective lines.

References

[1] M. F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. **85** (1957) 181–207.
[2] L. P. Eisenhart, *Non-Riemannian geometry*, Amer. Math. Soc. Colloq. Publ. 8, New York, 1927.

FACULTY OF MATHEMATICS, BUCHAREST 70109, RUMANIA